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New Method for Constraints Violation Suppression

Shlomo Djerassi*
RAFAEL, Ministry of Defense, 31021 Haifa, Israel
and
Keith W. Buffinton†
Bucknell University, Lewisburg, Pennsylvania 17837

The problem of constraints violation associated with numerical integration of dynamical equations of motion subject to constraints is dealt with. A new method for the suppression of constraints violation is presented. The method involves the definition of old constraints and of new constraints, the removal of the old constraints and the imposition of the new constraints, and the evaluation, after each integration step, of changes in the integration variables resulting from the removal–imposition process. The method is shown to be superior to the classical Baumgarte's method for constraints violation suppression in three aspects. First, the constraint violation measure number, a quantity defined herein, is smaller. Second, the constraint violation measure number stabilizes after the first integration step and does not degrade with time. Finally, the new method makes the process of trial and error, required by Baumgarte's method for the determination of integration constants, unnecessary. The method is discussed in connection with holonomic and simple nonholonomic constraints.

Introduction

CONSTRAINT violation is a long-standing problem associated with numerical integration of ordinary differential equations subject to constraints. Numerous investigations have been carried out in an attempt to alleviate the problem. References 1–6 form a nonexhaustive list of such attempts, which become even more cumbersome because different researchers use different integrators.

For systems subject to m constraints, Baumgarte[1] proposes the replacement of the sth constraint equation $\phi_s = 0\ (s = 1, \ldots, m)$ with $\delta_s\ddot{\phi}_s + 2\alpha_s\dot{\phi}_s + \beta_s^2\phi_s = 0$, where δ_s, α_s, and β_s are constants. For holonomic constraints $\delta_s = 1$, $\phi_s = \phi_s(q, t)$, and for simple nonholonomic constraints $\delta_s = 0$, $\phi_s = \phi_s(q, \dot{q}, t)$. Baumgarte[2] extends his method for holonomic constraints and recommends the replacement of $\phi_s = 0$ with $\dot{\phi}_s + 2\alpha_s\phi_s + \beta_s^2 \int \phi_s\, dt = 0$ in connection with motion equations using generalized coordinates and generalized momenta as variables. Rosen and Edelstien[3] extend Baumgarte's idea. They apply the variational principle to a Lagrangian including $\delta_s\ddot{\phi}_s + 2\alpha_s\dot{\phi}_s + \beta_s^2\phi_s$ and obtain modified sets of Lagrange's equations in addition to the modified constraint equations. Park and Chiou[4] replace $\phi_s = 0$ with the penalty form of the constraint equations $\phi_s = e\lambda_s$ (where e is an arbitrarily small constant and λ_s is the sth Lagrange multiplier) and solve the motion equations together with $d(\phi_s = e\lambda_s)/dt$. Kurdila et al.[5] extend Park's idea for holonomic systems, introducing the penalized potential and kinetic energies. These are defined $\dot{\phi}^T\beta\dot{\phi}/2$ and $\phi^T\alpha\phi/2$, respectively, where β and α are constant $m \times m$ positive definite matrices and $\phi \mathrel{\hat{=}} |\phi_1, \ldots, \phi_m|^T$, and, in the context of Lagrange's equations, augment the Lagrangian accordingly. Kurdila et al. prove convergence and stability of the method for certain classes of problems. Bayo et al.[6] take an approach similar to the one in Ref. 5; however, they suggest a formulation applicable to nonholonomic systems as well. Moreover, they develop an iterative procedure to obtain convergence, irrespective of the values of β and α.

All of these methods involve modified constraints and/or modified motion equations. Moreover, they involve the empirical determination of certain constants affecting the constraint violation measure number, a quantity defined in the sequel. Here, a new method for the suppression of constraint violation is introduced, leaving the motion equation as well as the constraint equations intact. The underlying idea of the new method can be discussed with the aid of the following example.

Consider a planar system comprising an elastic, uniform beam B of length L moving over supports represented by points $\bar{S}$ and $\hat{S}$ fixed in N, a Newtonian reference frame, as shown in Fig. 1. Let $\bar{P}$ and $\hat{P}$ be points of the neutral axis of B instantaneously in contact with $\bar{S}$ and $\hat{S}$, respectively; suppose that B is made to move over $\bar{S}$ and $\hat{S}$ by means of an actuator fixed to $\hat{S}$ and applying a driving force to B at $\hat{P}$.

A computer code based on equations governing motions of B was constructed and used in conjunction with a Cutta–Merson variable step integrator to simulate motions of B. The distances $z(0)$ and $z(L)$ of $\bar{B}$ and $\hat{B}$ (the endpoints of B) from U, i.e., the line passing through $\bar{S}$ and $\hat{S}$ (Fig. 1), were recorded for a motion starting from rest with $\hat{B}$ coinciding with $\hat{S}$. Figure 2 shows $z(0)$ as a function of time, indicating that $z(0)$ fails to converge to zero, in contradiction to that expected when $\bar{B}$ coincides with $\bar{S}$.

The convergence failure can be examined with the aid of $\boldsymbol{b}_N^{\bar{P}}$ and $\boldsymbol{b}_N^{\hat{P}}$, unit vectors perpendicular to the neutral axis of the beam at $\bar{P}$ and $\hat{P}$, and ${}^N\boldsymbol{v}^{\bar{P}}$ and ${}^N\boldsymbol{v}^{\hat{P}}$, the velocities of $\bar{P}$ and $\hat{P}$ in N, respectively. Ideally, the motion of B proceeds such that

$$
{}^N\boldsymbol{v}^{\bar{P}} \cdot \boldsymbol{b}_N^{\bar{P}} = 0, \qquad {}^N\boldsymbol{v}^{\hat{P}} \cdot \boldsymbol{b}_N^{\hat{P}} = 0
$$

However, evaluating the associated scalar functions in connection with numerical integration of the motion equations, one finds that

$$
{}^N\boldsymbol{v}^{\bar{P}} \cdot \boldsymbol{b}_N^{\bar{P}} = \bar{V}, \qquad {}^N\boldsymbol{v}^{\hat{P}} \cdot \boldsymbol{b}_N^{\hat{P}} = \hat{V}
$$

and that $\bar{V}$ and $\hat{V}$ become larger with time, indicating constraints violation.

Now, consider the following idea: The constraints in the second set of equations, called old constraints, are removed, and the constraints in the first set of equations, called new constraints, are imposed after each integration step. Furthermore, changes in the generalized speeds, the integration variables, resulting from constraints imposition removal are evaluated after each integration step. Applying this idea to the problem at hand one obtains Fig. 3, showing convergence of $z(0)$ to zero. In accordance with this idea, a new method for constraint violation suppression is set forth, merging Kane's equation for constrained systems[7] with the theory of imposition removal of constraints.[8] It is the purpose of this work to present the theory underlying the new method and apply it to a number of examples.

This paper is organized as follows. The main results of the theory of imposition and removal of constraints are reported in the following section. The new method is developed next, culminating in a

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*Chief Research and Development Engineer, P.O. Box 2250. Member AIAA.

†Associate Professor, Department of Mechanical Engineering.

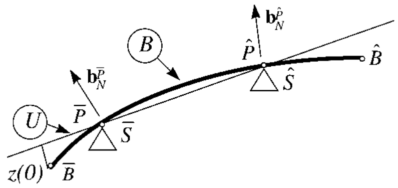


Fig. 1 Beam moving over fixed supports.

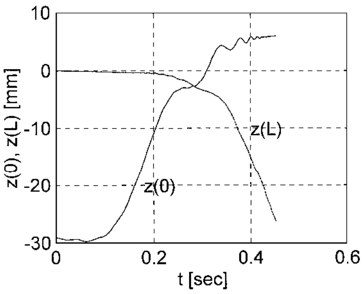


Fig. 2 Endpoints elastic deflection with $z(0)$ nonconvergence.

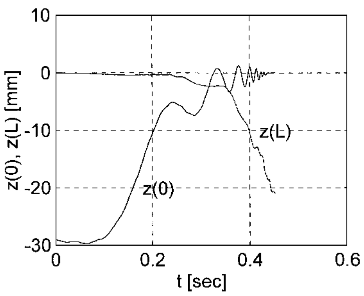


Fig. 3 Endpoints elastic deflection with $z(0)$ convergence (new algorithm).

set of linear, algebraic equations used to update the integration variables. The new method and Baumgarte's methods are applied to systems subject to holonomic and simple nonholonomic constraints, including the system shown in Fig. 1. A short discussion of both methods conclude the work.

New Method

Theory of Imposition Removal of Constraints: Main Results

The idea of imposition of constraints[7] comes to light in connection with a system S of v particles P_i $(i = 1, \dots, v)$ of mass m_i possessing $\bar{n}$ generalized coordinates $q_1, \dots, q_{\bar{n}}$ and n (where $n \le \bar{n}$) generalized speeds $u_1, \dots, u_n$ in N, a Newtonian reference frame, undergoing three phases of motion, as follows. Phase A occurs in the time interval $0 \le t \le t_1$. The motion of S in N is defined as unconstrained and governed by n dynamical equations, namely,

$$F_r + F_r^* = 0 \qquad (r = 1, \dots, n) \tag{1}$$

where F_r and F_r^* are the rth generalized active force and the rth generalized inertia force, respectively. Phase B, a transition phase, occurs in the time interval $t_1 \le t \le t_2$, where $t_2 - t_1$, is infinitely small, e.g., compared to time constants related to the motion of S. Then m constraints of the form

$$u_k = \sum_{r=1}^{p} C_{kr} u_r + D_k \qquad (k = p+1, \dots, n) \tag{2}$$

are imposed on S, where

$$p \mathrel{\hat{=}} n - m \tag{3}$$

and C_{kr} and D_k are functions of $q_1, \dots, q_{\bar{n}}$ and time t. In this phase, the configuration of S in N remains unaltered, that is,

$$q_r(t_2) = q_r(t_1) \qquad (r = 1, \dots, \bar{n}) \tag{4}$$

and the number of independent, generalized speeds is reduced from n to p. The relations between $u_k(t_2)$ $(k = p+1, \dots, n)$, the values of the dependent generalized speeds at $t = t_2$, and $u_r(t_2)$ $(r = 1, \dots, p)$, the values of the independent generalized speeds at $t = t_2$, are given by

$$u_k(t_2) = \sum_{r=1}^{p} C_{kr} u_r(t_2) + D_k \qquad (k = p+1, \dots, n) \tag{5}$$

Additionally, if the magnitudes of the active forces contributing to Eqs. (1) are all bounded, and if particles of S exert contact forces on one another and possibly on particles belonging to R_B, a set of rigid bodies with motion unaffected by the forces exerted on them by particles of S, then the relations between $u_s(t_2)$ $(s = 1, \dots, n)$ and $u_s(t_1)$ $(s = 1, \dots, n)$ are given by

$$\sum_{s=1}^{n} \left(m_{rs} + \sum_{k=p+1}^{n} C_{kr} m_{ks} \right) [u_s(t_2) - u_s(t_1)] = 0 \qquad (r = 1, \dots, p) \tag{6}$$

Here m_{rs}, the element in row r, column s of the mass matrix associated with Eqs. (1), is defined:

$$m_{rs} \mathrel{\hat{=}} -\sum_{i=1}^{v} m_i \frac{\partial \boldsymbol{v}^{P_i}}{\partial u_r} \cdot \frac{\partial \boldsymbol{v}^{P_i}}{\partial u_s} \tag{7}$$

where $\boldsymbol{v}^{P_i}$ is the velocity of P_i in N. These $m + p$ relations between $u_r(t_2)$ and $u_r(t_1)$ $(s = 1, \dots, n)$ enable the evaluation of the former, given the latter, with C_{kr}, D_k, and m_{rs} $(k = p+1, \dots, n;$ $r, s = 1, \dots, n)$ calculated at $t = t_1$. Phase C occurs when $t > t_2$. Then the motion of S in N is defined as constrained and governed by p dynamical equations, namely,

$$F_r + F_r^* + \sum_{k=p+1}^{n} C_{kr} \left(F_k + F_k^* \right) = 0 \qquad (r = 1, \dots, p) \tag{8}$$

Equations (8) are called Kane's equations of motion for constrained systems.

Removal of constraints occurs when the order of phases A, B, and C is reversed. The constrained phase (now called phase A), the transition, removal phase (phase B), and the unconstrained phase (now called phase C) occur when $0 \le t \le t_1, t_1 \le t \le t_2$, and $t > t_2$, and they are governed by Eqs. (8) and (2), (4–6), and (1), respectively, with t_1 replacing t_2 in Eqs. (5). If Eqs. (5) are satisfied both at t_1 and at t_2, then

$$u_r(t_2) = u_r(t_1) \qquad (r = 1, \dots, n) \tag{9}$$

Constraint Violation

Let

$$\dot{q}_r = \sum_{s=1}^{n} A_{rs} u_s + B_r \qquad (r = 1, \dots, \bar{n}) \tag{10}$$

be the kinematical equations associated with the motion of S in phase A [Eqs. (1)]. Let

$$\dot{q}_r = \sum_{s=1}^{p} X_{rs} u_s + Y_r \qquad (r = 1, \dots, \bar{n}) \tag{11}$$

be the kinematical equations associated with the motion of S in phase C [Eqs. (8)], where A_{rs}, B_r $(r = 1, \dots, \bar{n};$ $s = 1, \dots, n)$, and X_{rs}, Y_r $(r = 1, \dots, \bar{n};$ $s = 1, \dots, p)$ are functions of $q_1, \dots, q_{\bar{n}}$ and t. Equations (11) are obtained from Eqs. (10) if $u_{p+1}, \dots, u_n$ are eliminated with the aid of Eqs. (2). Moreover, if the time derivatives of both sides of Eqs. (2) are taken, then

$$\dot{u}_k = \sum_{r=1}^{p} (C_{kr} \dot{u}_r + \dot{C}_{kr} u_r) + \dot{D}_k \qquad (k = p+1, \dots, n) \tag{12}$$

Finally, define V_k $(k = p+1, \dots, n)$, called constraint violation measure number (CVMN), as

$$V_k \mathrel{\hat{=}} u_k - \left(\sum_{r=1}^{p} C_{kr} u_r + D_k \right) \qquad (k = p+1, \dots, n) \tag{13}$$

If $V_k = 0$ $(k = p+1, \ldots, n)$ throughout the integration, the system is said to be free of constraints violation. Otherwise, the motion constraints are violated and V_k $(k = p+1, \ldots, n)$ can be regarded as a measure of the constraint violation.

Next, consider the following approaches to the construction of equations governing motions of constrained systems.

1) The fact that the motion is constrained is temporarily disregarded, and n dynamical equations, namely, Eqs. (1), are formed, with $\dot{u}_1, \ldots, \dot{u}_n$ as unknowns. Moreover, Eqs. (2) and then Eqs. (12) are generated and used to eliminate $\dot{u}_{p+1}, \ldots, \dot{u}_n$ and $u_{p+1}, \ldots, u_n$ from Eqs. (1). The resulting equations are then recombined, as in Eqs. (8), and solved in conjunction with Eqs. (11) for $u_1, \ldots, u_p$ and $q_1, \ldots, q_{\bar{n}}$. Alternatively, the same set of equations can be obtained as follows. Expressions for velocities and angular velocities of points and of reference frames of interest are formed in terms of $u_1, \ldots, u_n$. Constraint equations are then formed and solved for $u_{p+1}, \ldots, u_n$, as in Eqs. (2). These are used to eliminate $u_{p+1}, \ldots, u_n$ from the indicated expressions. From here on, Kane's equations (Ref. 9, Sec. 6.1) are used to form p dynamical equations in p unknowns $\dot{u}_1, \ldots, \dot{u}_p$, and these, together with Eqs. (11), are solved for $u_1, \ldots, u_p, q_1, \ldots, q_{\bar{n}}$.

2) As before, the fact that the motion is constrained is temporarily disregarded; n dynamical equations, namely Eqs. (1), are formed with $\dot{u}_1, \ldots, \dot{u}_n$ as unknowns, and Eqs. (2) and Eqs. (12) are generated. Equations (1) are then reassembled as in Eqs. (8) and solved, together with Eqs. (12) and (10) for $u_1, \ldots, u_n$ and $q_1, \ldots, q_{\bar{n}}$.

Numerical errors associated with the integration of Eqs. (8) and (11) (first approach) or Eqs. (8), (10), and (12) (second approach) may lead to constraints violation, namely, to $V_k \neq 0$ $(k = p+1, \ldots, n)$. A new method for constraint violation suppression related to the second approach is discussed next.

New Method

Suppose that, in the context of the second approach, an integration step has been completed at time t_1 and that V_k $(k = p+1, \ldots, n)$ were evaluated and found to be different from zero. Then

$$u_k = \sum_{r=1}^{p} C_{kr} u_r + D_k + V_k \qquad (k = p+1, \ldots, n) \tag{14}$$

[see Eqs. (2)] are the constraint equations valid from t_1 on. These are called old constraint equations and imply that the values of u_k $(k = p+1, \ldots, n)$ used in the next integration step do not satisfy the exact constraint equations. The latter are given by Eqs. (2), namely,

$$u_k = \sum_{r=1}^{p} C_{kr} u_r + D_k \qquad (k = p+1, \ldots, n) \tag{15}$$

and are called new constraint equations. Thus, a desirable step would be to remove the old constraints in Eqs. (14), and impose the new constraints given by Eqs. (15). The removal of constraints leaves the values of integration variables intact, as indicated by Eqs. (9). However, imposition of constraints is accompanied, in general, by an instantaneous change in the values of the generalized speeds. These changes can be evaluated with the aid of Eqs. (6) as follows.

Suppose $F_r + F_r^*$ $(r = 1, \ldots, n)$ have been formulated for S, which is temporarily regarded as undergoing an unconstrained motion. Then, in view of Eqs. (1), (2), and (8), the equations governing the motion of S in N from t_1 on are given by

$$\left(F_r + F_r^*\right)\Big|^{O} + \sum_{k=p+1}^{n} C_{kr}\left(F_k + F_k^*\right)\Big|^{O} = 0 \qquad (r = 1, \ldots, p) \tag{16}$$

where the superscript O indicates that, in the context of numerical integration of these equations, the values of generalized speeds satisfying Eqs. (14), the old constraint equations, are used. Moreover,

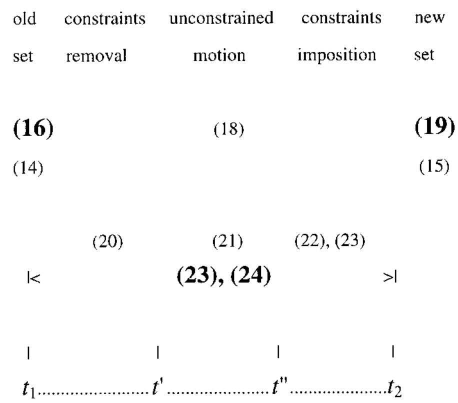


Fig. 4 Equations describing the state of affairs at end of each Δt.

suppose that three points in time t', t'', and t_2 are introduced such that

$$t_1 < t' < t'' < t_2 \tag{17}$$

as shown in Fig. 4, where $t' - t_1$, $t'' - t'$, and $t_2 - t''$ are infinitely small. Finally, assume that the constraints in Eqs. (14) are removed between t_1 and t', that between t' and t'' the motion of S in N is unconstrained and, hence, governed by the equations

$$F_r + F_r^* = 0 \qquad (r = 1, \ldots, n) \tag{18}$$

and that between t'' and t_2 the constraints in Eqs. (15) are imposed on S. This means that after t_2 the governing equations are

$$\left(F_r + F_r^*\right)\Big|^{N} + \sum_{k=p+1}^{n} C_{kr}\left(F_k + F_k^*\right)\Big|^{N} = 0 \qquad (r = 1, \ldots, p) \tag{19}$$

where values of u_k $(k = p+1, \ldots, n)$ satisfying Eqs. (15), the new constraint equations, are used, hence the superscript N.

Letting Eqs. (16), (18), and (14) play the roles of Eqs. (8), (1), and (2), respectively, in connection with the removal of constraints occurring between t_1 and t', one may conclude in view of Eqs. (9) that

$$u_r(t') = u_r(t_1) \qquad (r = 1, \ldots, n) \tag{20}$$

where t' plays the role of t_2 in Eq. (9). Next, Eqs. (18) have to be integrated from t' to t''. However, $t'' - t'$ is infinitely small, so that Eqs. (18) need not be actually integrated, and

$$u_r(t'') = u_r(t') \qquad (r = 1, \ldots, n) \tag{21}$$

Furthermore, Eqs. (18), (19), and (15) play the role of Eqs. (1), (8), and (2) in connection with the imposition of constraints occurring between t'' and t_2. It may be concluded, in view of Eqs. (6) and (5), that

$$\sum_{s=1}^{n}\left(m_{rs} + \sum_{k=p+1}^{n} C_{kr} m_{ks}\right)[u_s(t_2) - u_s(t'')] = 0 \qquad (r = 1, \ldots, p) \tag{22}$$

$$u_k(t_2) = \sum_{r=1}^{p} C_{kr} u_r(t_2) + D_k \qquad (k = p+1, \ldots, n) \tag{23}$$

where t'' in Eq. (22) plays the role to t_1 in Eq. (6). Equations (20), (21), and (22) can be replaced with

$$\sum_{s=1}^{n}\left(m_{rs} + \sum_{k=p+1}^{n} C_{kr} m_{ks}\right)[u_s(t_2) - u_s(t_1)] = 0 \qquad (r = 1, \ldots, p) \tag{24}$$

These, together with Eqs. (23), form a set of linear, algebraic equations, which, in connection with the new method, are solved for

$u_1(t_2), \ldots, u_n(t_2)$ in terms of $u_1(t_1), \ldots, u_n(t_1)$ after each integration step. Thus, $u_1(t_2), \ldots, u_n(t_2)$ are the values of the generalized speeds used at the next integration step.

Examples and Discussion

Of all available methods for constraints violation suppression, Baumgarte's method is the most widely used both in actual application[10,11] and as a referenced method.[3,4,6] Used here as a reference, the method involves the replacement of Eqs. (12) with the following equations:

$$\dot{V}_k + 2\alpha_k V_k + \beta_k^2 \int V_k \, dt = 0 \qquad (k = p+1, \ldots, n) \qquad (25)$$

where α_k and β_k $(k = p+1, \ldots, n)$ are constants, as indicated in the Introduction.

Consider, for example, a planar, four-bar mechanism comprising uniform links A, B, C, and D (the latter is fixed in N), of lengths l_A, l_B, l_C, and l_D, respectively, and equal mass per unit length ρ. Suppose q_1, q_2, and q_3 are angles describing the orientation of A, B, and C in N, as shown in Fig. 5. Furthermore, let ${}^N\boldsymbol{v}^{\hat{C}}$ be the velocity in N of $\hat{C}$, an endpoint of C, noting that the motion proceeds so that ${}^N\boldsymbol{v}^{\hat{C}} = \boldsymbol{0}$ and suppose that the constraints that force ${}^N\boldsymbol{v}^{\hat{C}}$ to vanish are temporarily removed. Then q_1, q_2, and q_3 become independent, and three motion equations can be written, the first being

$$-3.33J\dot{u}_1 - 4J\cos(q_1 - q_2)\dot{u}_2 - 0.5J\cos(q_1 - q_3)\dot{u}_3$$
$$-0.5J\left[\sin(q_1 - q_3)u_3^2 + 8\sin(q_1 - q_2)u_2^2\right] = 0 \qquad (26)$$

where $u_r = \dot{q}_r$ $(r = 1, 2, 3)$, $J = \rho l^3$, and $l_A = l$, $l_B = 2l$, $l_C = l$, and $l_D = 3l$. Substitutions in ${}^N\boldsymbol{v}^{\hat{C}} = \boldsymbol{0}$ lead to two constraint equations, which, if cast in the form of Eqs. (2), read

$$u_2 = C_{21}u_1, \qquad C_{21} \mathrel{\hat{=}} \frac{-\sin(q_1 - q_3)}{2\sin(q_2 - q_3)} \qquad (27)$$

$$u_3 = C_{31}u_1, \qquad C_{31} \mathrel{\hat{=}} \frac{\sin(q_1 - q_2)}{\sin(q_2 - q_3)} \qquad (28)$$

One can use Eqs. (8) with $n = 3$, $m = 2$, and $p = 1$ to generate the governing dynamical equation, and integrate this equation together with the equations

$$\dot{u}_r = C_{r1}\dot{u}_1 + \dot{C}_{r1}u_r \qquad (r = 2, 3) \qquad (29)$$

obtained from Eqs. (27) and (28).

Let $l = 0.2$ m and $\rho = 5$ kg/m and consider a simulation starting with $q_1 = 60$, $q_2 = 0$, $q_3 = -60$ deg, and $u_1 = -0.1$ rad/s. Under these circumstances $V_2 \cong V_3 \cong 10^{-4}$ after 1000 s, whereas initially $V_2(0) \cong V_3(0) \cong 10^{-18}$. It should be noted that each of $q_1 - q_2$, $q_1 - q_3$, and $q_2 - q_3$ vanish every 30 s, a fact underlying the deterioration in the CVMNs. Baumgarte's method with $\alpha_1 = \alpha_2 = -0.2$ and $\beta_1 = \beta_2 = 0$ leads to a slight improvement, namely, to $V_2 \cong V_3 \cong 10^{-5}$. (Numerical experiments show that β_1 and $\beta_2 \neq 0$ increase the CVMNs.)

The new method requires the updating of u_1, u_2, and u_3 after each integration step, as follows:

$$u_1(t_2) = \sum_{r=1}^{3} \frac{m_r u_r(t_1)}{m_1 + C_{21}m_2 + C_{31}m_3}$$
$$m_r = m_{1r} + C_{21}m_{2r} + C_{31}m_{3r} \qquad (30)$$
$$u_r(t_2) = C_{r1}u_1(t_2) \qquad (r = 2, 3)$$

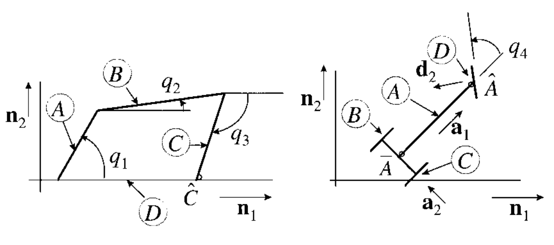


Fig. 5 Four-bar linkage and a tricycle.

obtained by substitutions in Eqs. (23) and (24) (for $n = 3$ and $p = 1$), where m_{rs} $(r, s = 1, 2, 3)$ are the elements of the mass matrix associated with Eq. (26) and C_{21} and C_{31} are as in Eqs. (27) and (28). One then has $V_2 \cong V_3 \cong 10^{-18}$ throughout the motion. It is worth looking at the functions $f_1 = l_A\cos(q_1) + l_B\cos(q_2) + l_C\cos(q_3)$ and $f_2 = \text{KE}(u_1, u_2, u_3, q_1, q_2, q_3)$. The former equals l_D, and the latter composes the kinetic energy of the system. Here $l_D = 0.6$ m and $\text{KE}(0) = 4.8 \times 10^{-4}$ N-m, and f_1 and f_2 are supposed to equal l_D and KE(0), respectively, throughout the integration. However, constraint violation leads, after 1000 s, to $f_1 = 0.5827$ m and $f_2 = 4.490 \times 10^{-4}$ N-m. Baumgarte's method yields $f_1 = 0.5914$ m and $f_2 = 4.647 \times 10^{-4}$ N-m, whereas the new method results in $f_1 = 0.5999997$ m and $f_2 = 4.79995 \times 10^{-4}$ N-m. One may conclude that, although the new method updates only motion variables, i.e., generalized speeds, the CVMNs are small enough to prevent significant errors in configuration variables, i.e., generalized coordinates.

Note that the elimination of u_2 and u_3 (first approach) leads to $f_1 = 0.6$ m and $f_2 = 2.639 \times 10^{-4}$ N-m, results indicating constraint violation.

Consider next the tricycle S shown in Fig. 5, whose motion proceeds in a plane perpendicular to a unit vector $\boldsymbol{n}_3$ and involves nonholonomic constraints. The tricycle consists of body A of mass m_A and central moment of inertia I_A for $\boldsymbol{n}_3$, and three massless wheels B, C, and D. If u_r $(r = 1, 2, 3)$ are defined such that ${}^N\boldsymbol{v}^{A^*} = u_1\boldsymbol{a}_1 + u_2\boldsymbol{a}_2$ and ${}^N\omega^A = u_3\boldsymbol{n}_3$, where A^* is the mass center of A, and ${}^N\boldsymbol{v}^{A^*}$ and ${}^N\omega^A$ are the velocity of A^* in N and the angular velocity of A in N, then the equations of motion of S are

$$m_A(-\dot{u}_1 + u_2u_3) + F = 0$$
$$m_A(\dot{u}_2 + u_1u_3) = 0, \qquad -I_A\dot{u}_3 = 0 \qquad (31)$$

if the motion is unconstrained, where $F \mathrel{\hat{=}} \boldsymbol{F} \cdot \boldsymbol{a}_1$, $\boldsymbol{F}$ being the force applied at $\bar{A}$. Now, suppose that the interaction between the wheels and the ground does not allow side slip of the wheels. Then

$${}^N\boldsymbol{v}^{\bar{A}} \cdot \boldsymbol{a}_2 = 0, \qquad {}^N\boldsymbol{v}^{\hat{A}} \cdot \boldsymbol{d}_2 = 0 \qquad (32)$$

where $\bar{A}$ and $\hat{A}$ are points of A, and $\boldsymbol{n}_i$, $\boldsymbol{a}_i$, and $\boldsymbol{d}_i$ $(i = 1, 2, 3)$ are three sets of dextral, mutually perpendicular unit vectors fixed in N, A, and D, respectively, as shown in Fig. 5. Expressing ${}^N\boldsymbol{v}^{\bar{A}}$ and ${}^N\boldsymbol{v}^{\hat{A}}$ in terms of u_r $(r = 1, 2, 3)$, one obtains, using Eqs. (32),

$$u_2 = C_{21}u_1, \qquad C_{21} \mathrel{\hat{=}} 0.5\tan(q_4)$$
$$u_3 = C_{31}u_1 \qquad C_{31} \mathrel{\hat{=}} \frac{0.5\tan(q_4)}{L} \qquad (33)$$

where q_4 is the angle shown in Fig. 5 and L is the distance from $\bar{A}$ to $\hat{A}$. Substitutions in Eqs. (8) give rise to a governing equation ($p = 1$), which can be integrated without appreciable constraint violation if u_r $(r = 1, 2, 3)$ satisfy Eqs. (33) at $t = 0$. Suppose, however, that $u_1(0) = 59.70$ m/s, $u_2(0) = 6.1$ m/s, and $u_3(0) = 6.1$ rad/s, values that, with $q_4 = 0.1984$ rad, slightly violate Eqs. (33). Integration of the indicated equations with $L = 1$ m, $F = 10$ N, $m_A = 0.01$ kg, and $I_A = 0.25$ kg-m^2 reveals that $V_2 = V_3 = 0.09881$ throughout the motion. Use of Baumgarte's procedure with $\alpha_2 = 500$ and $\alpha_3 = 5$ lead, after 120 s, to $V_2 \cong 10^{-6}$ and $V_3 \cong 10^{-15}$.

As before, the new method involves the updating of u_r $(r = 1, 2, 3)$ after each integration step. Here, $n = 3$, $m = 2$, and $p = 1$; hence, Eqs. (30) are valid and, written explicitly, read

$$u_1(t_2) = \frac{m_A + C_{21}m_Au_2(t_1) + C_{31}I_Au_3(t_1)}{(m_A + C_{21}m_A + C_{31}I_A)}$$
$$u_2(t_2) = C_{21}u_1(t_2), \qquad u_3(t_2) = C_{31}u_1(t_2) \qquad (34)$$

With these one obtains $V_2 \cong V_3 \cong 10^{-15}$. Moreover, with Baumgarte's procedure it takes 4 s (simulated time) to reach the indicated values of the CVMNs, whereas with the new procedure the indicated values are obtained from the start. Finally, the simulation runtime is 10 s with the new method as compared with 30 s with Baumgarte's method.

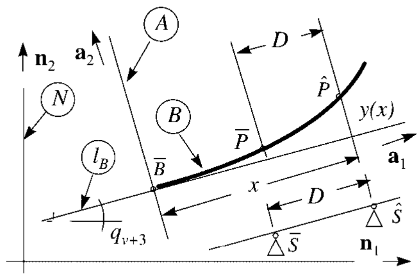


Fig. 6 Beam in an unconstrained motion.

Last, consider Fig. 6, which shows the beam of Fig. 1 in detail in its unconstrained configuration. The motion of the beam is constrained in a manner that can be described as follows. Let l_B be a straight line tangent to the neutral axis of B at $\bar{B}$ at all times. Let A be a reference frame and $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, and $\boldsymbol{a}_3$ be three dextral, mutually perpendicular unit vectors fixed in A and oriented so that $\boldsymbol{a}_1$ is parallel to l_B, and let $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ be parallel to the plane in which B moves. Accordingly, $\bar{B}$ is fixed in A and, if $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, and $\boldsymbol{n}_3$ are three dextral, mutually perpendicular unit vectors fixed in N, then A is oriented in N so that the angle between $\boldsymbol{a}_1$ and $\boldsymbol{n}_1$ is q_{v+3}, as shown in Fig. 6, where v is a number to be defined presently.

Now $\boldsymbol{p}^{\bar{B}\bar{P}}$ and $\boldsymbol{p}^{\bar{B}\hat{P}}$, the position vectors from $\bar{B}$ to $\bar{P}$ and $\hat{P}$, respectively, are given by

$$\boldsymbol{p}^{\bar{B}\bar{P}} = (x - D)\boldsymbol{a}_1 + y(x - D)\boldsymbol{a}_2, \qquad \boldsymbol{p}^{\bar{B}\hat{P}} = x\boldsymbol{a}_1 + y(x)\boldsymbol{a}_2 \tag{35}$$

where $y(x)$ and $y(x - D)$ are the elastic deflections at $\bar{P}$ and $\hat{P}$, respectively. Moreover, the assumed mode method makes it possible to describe the elastic deformation of B at $\hat{P}$ as

$$y(x) = \sum_{j=1}^{v} \phi_j(x)q_j(t) \tag{36}$$

Here, $\phi_j(x)$ is a function of x called the jth modal function, $q_j(t)$ is a function of time t called the jth modal coordinate, and v is the number of modes used to describe $y(x)$. One may associate with the unconstrained motion of S in N generalized speeds defined as follows:

$$u_r \hat{=} \dot{q}_r \qquad (r = 1, \ldots, v)$$
$$u_{v+r} \hat{=} {}^N\boldsymbol{v}^{\bar{B}} \cdot \boldsymbol{n}_r, \qquad u_{v+3} \hat{=} {}^N\omega^A \cdot \boldsymbol{n}_3 \qquad (r = 1, 2) \tag{37}$$

where ${}^N\boldsymbol{v}^{\bar{B}}$ and ${}^N\omega^A$ are, respectively, the velocity of $\bar{B}$ in N and the angular velocity of B in N. Hence,

$${}^N\boldsymbol{v}^{\bar{B}} = u_{v+1}\boldsymbol{n}_1 + u_{v+2}\boldsymbol{n}_2, \qquad {}^N\omega^A = u_{v+3}\boldsymbol{n}_3 \tag{38}$$

$$\dot{y}(x) = \sum_{j=1}^{v} \phi_j(x)u_j(t), \qquad y'(x) = \sum_{j=1}^{v} \phi'_j(x)q_j(t) \tag{39}$$

where $(\dot{\ })$ and $(\ )'$ indicate differentiation with respect to t and x, respectively. If q_{v+1} and q_{v+2} are the Cartesian coordinates of $\bar{B}$ in N, and q_{v+3} is the angle between $\boldsymbol{a}_1$ and $\boldsymbol{n}_1$, then

$$\dot{q}_{v+r} = u_{v+r} \qquad (r = 1, 2, 3) \tag{40}$$

The velocities of $\bar{P}$ and $\hat{P}$ in N can be expressed, in view of Eqs. (35) and (38), as

$${}^N\boldsymbol{v}^{\bar{P}} = u_{v+1}\boldsymbol{n}_1 + u_{v+2}\boldsymbol{n}_2 + [(x - D)u_{v+3} + \dot{y}(x - D)]\boldsymbol{a}_2 - y(x - D)u_{v+3}\boldsymbol{a}_1 \tag{41}$$

$${}^N\boldsymbol{v}^{\hat{P}} = u_{v+1}\boldsymbol{n}_1 + u_{v+2}\boldsymbol{n}_2 + [xu_{v+3} + \dot{y}(x)]\boldsymbol{a}_2 - y(x)u_{v+3}\boldsymbol{a}_1 \tag{42}$$

If, in addition, $\boldsymbol{b}_N^{\bar{P}}$ and $\boldsymbol{b}_N^{\hat{P}}$ are expressed as

$$\boldsymbol{b}_N^{\bar{P}} = -\sin[y'(x - D)]\boldsymbol{a}_1 + \cos[y'(x - D)]\boldsymbol{a}_2 \tag{43}$$

$$\boldsymbol{b}_N^{\hat{P}} = -\sin[y'(x)]\boldsymbol{a}_1 + \cos[y'(x)]\boldsymbol{a}_2 \tag{44}$$

then substitutions from Eqs. (41–44) and (39) in the first set of equations displayed in the Introduction lead to two equations, linear in u_r $(r = 1, \ldots, v + 3)$, which can be solved for u_{v+2} and u_{v+3}. Furthermore, if one regards u_{v+2} and u_{v+3} as functions of q_r $(r = 1, \ldots, v + 3)$ and u_r $(r = 1, \ldots, v + 1)$, and linearizes u_{v+2} and u_{v+3} in q_r and u_r $(r = 1, \ldots, v)$ about $q_r = u_r = 0$ $(r = 1, \ldots, v)$, then one obtains

$$u_{v+2} = -\sum_{j=1}^{v} \frac{[x\phi_j(x - D) - (x - D)\phi_j(x)]u_j}{D\cos q_{n+3}} + \left\{\tan q_{v+3} + \frac{[xy'(x - D) - (x - D)y'(x)]}{D\cos^2 q_{v+3}}\right\}u_{v+1} \tag{45}$$

$$u_{v+3} = \sum_{j=1}^{v} \frac{[\phi_j(x - D) - \phi_j(x)]u_j}{D} - \frac{[y'(x - D) - y'(x)]u_{v+1}}{D\cos q_{v+3}} \tag{46}$$

equations which are cast in the form of Eqs. (2).

Completion of the kinematics of this problem involves the evaluation of $\dot{x}$. To this end, one may regard x as the $\boldsymbol{a}_1$ component of the position vector from $\bar{B}$ to $\hat{S}$ and note that

$${}^A\boldsymbol{v}^{\hat{P}/\hat{S}} = \dot{x}\boldsymbol{b}_T^{\hat{P}}, \qquad \boldsymbol{b}_T^{\hat{P}} = \cos[y'(x)]\boldsymbol{a}_1 + \sin[y'(x)]\boldsymbol{a}_2 \tag{47}$$

where $\boldsymbol{b}_T^{\hat{P}}$ is a unit vector tangent to the neutral axis of B at $\hat{P}$ and ${}^A\boldsymbol{v}^{\hat{P}/\hat{S}}$ is the velocity of $\hat{P}$ relative to $\hat{S}$ in A. Because $\hat{P}$ and $\hat{S}$ coincide momentarily, $\boldsymbol{p}^{\hat{P}\hat{S}}$, the position vector from $\hat{P}$ to $\hat{S}$, vanishes. One can next show that[12] ${}^N\boldsymbol{v}^{\hat{P}} \cdot \boldsymbol{b}_T^{\hat{P}} + \dot{x} = 0$. Writing this expression explicitly and linearizing $\dot{x}$ in the same variables as before, one arrives at

$$\dot{x} = -u_{v+1}\cos q_{v+1} - u_{v+2}\sin q_{v+1} + y'(x)(u_{v+1}\sin q_{v+1} - u_{v+2}\cos q_{v+1}) + [y(x) - xy'(x)]u_{v+3} \tag{48}$$

Note that if x is defined as q_{v+4} (and $\dot{x}$ as u_{v+4}), then Eq. (48) can be regarded as an additional kinematical equation.

Next, $F_r + F_r^*$ $(r = 1, \ldots, v + 3)$, associated with the motion of B in N regarded as unconstrained, are generated. One may start with the assumption that the elastic deformation of B is adequately described by the Bernoulli–Euler beam theory, and consider the contribution to F_r^* of $\mathrm{d}B$, an element of B of length $\mathrm{d}\xi$ whose midpoint P is located at $\xi\boldsymbol{a}_1 + y(\xi)\boldsymbol{a}_2$ relative to $\bar{B}$. The velocity of P in N, ${}^N\boldsymbol{v}^P$, given by an expression similar to Eq. (42), with ξ replacing x, can be used to construct the requisite contributions $F_r^{*\mathrm{d}B}$ $(r = 1, \ldots, v + 3)$. These are functions of ξ and can be integrated from 0 to L, yielding expressions for F_r^* $(r = 1, \ldots, v + 3)$ for the entire beam. Expressions for F_r $(r = 1, \ldots, v + 3)$ include contributions from the following forces: 1) gravity forces, which, in connection with $\mathrm{d}B$, can be expressed as $-\rho\,\mathrm{d}\xi g\boldsymbol{n}_2$, where ρ is the mass per unit length of B, and g is the gravitational acceleration; 2) driving force $\boldsymbol{F}$, exerted by $\hat{S}$ on $\hat{P}$, and given by

$$\boldsymbol{F} = F\boldsymbol{b}_T^{\hat{P}} \tag{49}$$

where F is a constant; and 3) elastic forces, which, for a uniform beam, are associated with the following strain energy function, namely,

$$\frac{1}{2}EJL\sum_{j=1}^{v} q_j^2\lambda_j^4$$

where EJ is the bending rigidity of B and λ_j is the eigenvalue associated with $\phi_j(x)$ and q_j. With E_j, F_j, and $Z_1, \ldots, Z_6$ defined as

$$E_j \hat{=} L^{-1}\int_0^L \phi_j(\xi)\,\mathrm{d}\xi, \qquad F_j \hat{=} L^{-2}\int_0^L \xi\phi_j(\xi)\,\mathrm{d}\xi \qquad (j = 1, \ldots, v) \tag{50}$$

$$Z_1 \hat{=} \sum_{j=1}^{v}\phi_j(L)q_j, \qquad Z_2 \hat{=} \sum_{j=1}^{v}\phi_j(L)u_j, \qquad Z_3 \hat{=} \sum_{j=1}^{v}E_j q_j$$

$$Z_4 \hat{=} \sum_{j=1}^{v}E_j u_j, \qquad Z_5 \hat{=} \sum_{j=1}^{v}q_j u_j, \qquad Z_6 \hat{=} \sum_{j=1}^{v}q_j^2$$

one can show that $F_r + F_r^*$ $(r = 1, \ldots, v + 3)$ read

$$F_j + F_j^* = M\big[\dot{u}_{v+1}s_{v+3}E_j - \dot{u}_{v+2}c_{v+3}E_j - LF_j\dot{u}_{v+3} - \dot{u}_j + u_{v+3}^2 q_j\big] - EJLq_j\lambda_j^4 - c_{v+3}E_j Mg + F\phi_j(x)y'(x) \qquad (j = 1, \ldots, v) \tag{51}$$

$$F_{v+1} + F_{v+1}^* = M\bigg[-\dot{u}_{v+1} + \frac{L}{2}\dot{u}_{v+3}s_{v+3} + \frac{L}{2}u_{v+3}^2 c_{v+3} + s_{v+3}\sum_{j=1}^{v}E_j\dot{u}_j + 2u_{v+3}c_{v+3}Z_4 + \big(\dot{u}_{v+3}c_{v+3} - u_{v+3}^2 s_{v+3}\big)Z_3\bigg] + F[c_{v+3} - y'(x)s_{v+3}] \tag{52}$$

$$F_{v+2} + F_{v+2}^* = M\bigg[-\dot{u}_{v+2} - \frac{L}{2}\dot{u}_{v+3}c_{v+3} + \frac{L}{2}u_{v+3}^2 s_{v+3} - c_{v+3}\sum_{j=1}^{v}E_j\dot{u}_j + 2u_{v+3}s_{v+3}Z_4 + \big(\dot{u}_{v+3}s_{v+3} + u_{v+3}^2 c_{v+3}\big)Z_3\bigg] - Mg + F[s_{v+3} + y'(x)c_{v+3}] \tag{53}$$

$$F_{v+3} + F_{v+3}^* = M\bigg[\frac{L}{2}\dot{u}_{v+1}s_{v+3} - \frac{L}{2}\dot{u}_{v+2}c_{v+3} - \frac{L^2}{3}\dot{u}_{v+3} - L\sum_{j=1}^{v}F_j\dot{u}_j + \dot{u}_{v+1}c_{v+3}Z_3 + \dot{u}_{v+2}s_{v+3}Z_3 - 2u_{v+3}Z_5 - \dot{u}_{v+3}Z_6\bigg] - Mg\left(\frac{L}{2}c_{v+3} - s_{v+3}Z_3\right) + F[-y(x) + xy'(x)] \tag{54}$$

where $s_{v+3} \hat{=} \sin q_{v+3}$, $c_{v+3} \hat{=} \cos q_{v+3}$, and $M = \rho l$. Moreover, in accordance with the new method, Eqs. (45) and (46) play the role of Eqs. (23) if u_r $(r = 1, \ldots, v + 3)$ are replaced with $u_r(t_2)$ $(r = 1, \ldots, v + 3)$, and the role of Eqs. (24) is played by the matrix equation obtained by substitution in Eqs. (24) of $C_{v+2,j}$ and $C_{v+3,j}$ $(j = 1, \ldots, v+1)$, the coefficients of u_j in Eqs. (45) and (46), and of m_{rs} $(r, s = 1, \ldots, v + 3)$ from Eqs. (51–54) (Ref. 12). Thus, $u_1(t_2), \ldots, u_{v+3}(t_2)$ can be evaluated.

Last, initial conditions must be determined. Here, a choice was made to let the motion start from rest; then $u_r(0) = 0$ $(r = 1, \ldots, v+3)$. Moreover, $q_1(0), \ldots, q_{v+3}(0)$ and $F(0)$ are chosen to have values associated with the static deflection of B if acted upon by gravity in the $-\boldsymbol{n}_2$ direction. To obtain these values, one may take the following steps. First, substitute from Eqs. (51–54) and (45) and (46) in Eqs. (8), then set $u_r = \dot{u}_r = 0$ $(r = 1, \ldots, v+3)$, and solve the $v + 1$ resulting equations together with the following three equations[12]:

$$q_{v+1}(0) = x_0[\cos\alpha - \cos q_{v+3}(0)] + y(x_0, 0)\sin q_{v+3}(0) \tag{55}$$

$$q_{v+2}(0) = x_0[\sin\alpha - \sin q_{v+3}(0)] - y(x_0, 0)\cos q_{v+3}(0) \tag{56}$$

$$q_{v+3}(0) = \frac{\alpha - [y(x_0, 0) - y(x_0 - D, 0)]}{D} \tag{57}$$

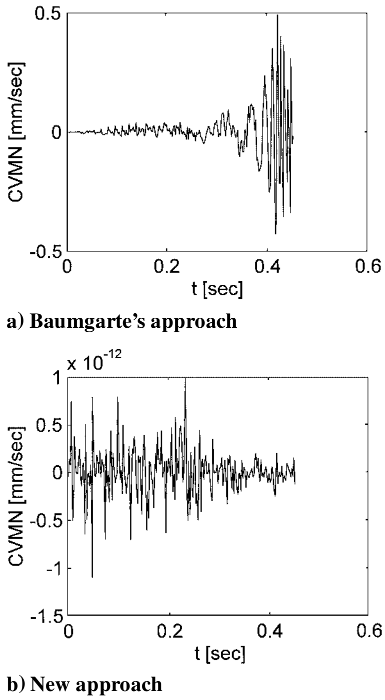
for $q_1(0), \ldots, q_{v+3}(0)$ and $F(0)$.

One is now in a position to implement the new method in conjunction with Eqs. (51–54) and (45) and (46), used to construct equations playing the role of Eqs. (16), (23), (24), and (19). Simulating the motion of S with a code based on these equations with the following numerical values: $v = 6$, $EJ = 0.084$ kg-m^2, $M = 0.036$ kg, $L = 1$ m, $D = 0.2$ m, $\alpha = 0.2$ rad, and $F = 0.36$ kg, and with $u_r(0) = 0$ $(r = 1, \ldots, v + 3)$, one obtains Fig. 3 with $V_8 \cong V_9 \cong 10^{-14}$ throughout the motion (see Fig. 7). Disregarding Eqs. (23) and (24), on the other hand, one obtains Fig. 2. The associated CVMNs deteriorates gradually and become $V_8(t_f) \cong 10$ and $V_9(t_f) \cong 10^{-3}$, t_f being the time $\bar{B}$ coincides with $\bar{S}$.

In connection with Baumgarte's method, it should be noted that constraint equations have been generated as if the constraints were simple nonholonomic [and that, as a result of the linearization leading to Eqs. (45) and (46), there are no relations between the generalized coordinates with a time derivative leading to these equations; hence, $\beta_8 = \beta_9 = 0$]. Integrating Eqs. (51–54) together with the time derivatives of Eqs. (45) and (46) and choosing $\alpha_8 = 50{,}000$ and $\alpha_9 = 5000$, one obtains results closely resembling those in Fig. 3; however, $V_8(t_f) \cong 10^{-1}$ and $V_9(t_f) \cong 10^{-2}$ as shown in Fig. 7. The CVMNs become larger with time and cannot be reduced with a different choice of α_8 and α_9 (values of these constants larger than the ones indicated slow down the integration prohibitively).

It is finally worth noting that if the dependent variables are eliminated with the aid of the constraint equations, one obtains results similar to those in Fig. 3.

In summary, the various examples show that the CVMNs associated with the new method are always smaller than those obtained with Baumgarte's method. Moreover, the CVMNs stabilize after the first integration step, remain stable throughout the integration, and are small enough to allow no propagation of configuration errors. No integration constants and, hence, no trial-and-error process is involved. Such a process is required by each of the methods referred to in the Introduction to determine values of integration constants. Precisely the same formulation of the new method applies to both



a) Baumgarte's approach

b) New approach

Fig. 7 Constraints violation.

holonomic and simple nonholonomic systems. Finally, the method has no impact on the simulation efficiency, i.e., runtime.

Conclusions

A new method for constraints violation suppression is presented in connection with the numerical integration of motion equations subject to constraints. The method is based on a logical extension of the discretization associated with numerical integration of the indicated equations. The underlying idea is that at each integration step violated, old constraints are removed and exact, new constraints are imposed, and the associated change in the integration variables are accounted for. The method was applied to a number of examples and was shown, in connection with these examples, to be superior to the classical Baumgarte's method. Future work involves a comparative study of methods for constraint violation suppression when applied to equations of motion of systems subject to holonomic and simple nonholonomic constraints, integrated by a variety of numerical integrators.

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